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On periodic boundary value problem for the Sturm-Liouville operator

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We consider the eigenvalue problem for the Sturm-Liouville operator

$$Lu = u'' - q(x)u \tag{1}$$

with boundary conditions

$$u(0) \mp u(1) = 0, \quad u'(0) \mp u'(1) = 0 \tag{2}$$

where either the sign "$-$" is chosen two times (case 1) or the sign "$+$" is chosen two times (case 2), i.e. boundary conditions (2) are periodic or antiperiodic. Function $q(x)$ is an arbitrary complex-valued function of the class $L_1(0,1)$.

Let $\{u_n(x)\}$ be the system of eigenfunctions and associated functions of problem (1)+(2). It is well known that this system is complete and minimal in $L_2(0,1)$. We denote

$$\alpha_n = \int_0^1 q(x)e^{2\pi inx}dx, \quad \beta_n = \int_0^1 q(x)e^{-2\pi inx}dx.$$

Suppose the function $q(x)$ satisfies the following conditions: $q(x) \in W_1^m[0,1]$, $q^{(j)}(0) = q^{(j)}(1)$ where $j = \overline{0, m-1}$, $m = 0, 1, \ldots$.

Theorem 1. *If for all even (in the case 1) or odd (in the case 2) $n > n_0$ where n_0 is a natural number*

$$|\alpha_n| > \frac{c_0}{n^{m+1}}, \quad 0 < c_1 < |\frac{\alpha_n}{\beta_n}| < c_2$$

($c_0 > 0$), *then the root function system $\{u_n(x)\}$ of corresponding problem (1)+(2) forms a Riesz basis for $L_2(0,1)$.*

Theorem 2. *If there exists a sequence of even (in the case 1) or odd (in the case 2) numbers n_k ($k = 1, 2, \ldots$) such that*

$$|\alpha_{n_k}| > \frac{c_0}{n_k^{m+1}}, \quad |\beta_{n_k}| > \frac{c_0}{n_k^{m+1}}$$

($c_0 > 0$) *moreover $\lim_{k\to\infty}(|\alpha_{n_k}/\beta_{n_k}| + |\beta_{n_k}/\alpha_{n_k}|) = \infty$, then the root function system $\{u_n(x)\}$ of corresponding problem (1)+(2) is not a basis for $L_2(0,1)$.*

It is easy to verify that the function

$$q(x) = \sum_{n=1}^{\infty} \gamma_n \left(\frac{e^{2\pi inx}}{n^{\varepsilon_1}} + \frac{e^{-2\pi inx}}{n^{\varepsilon_2}} \right)$$

satisfies all conditions of Theorem 2. Here $0 < \varepsilon_1 < \varepsilon_2 < 1$ and also in the case 1 $\gamma_n = 1$ if $n = 2^p$ and $\gamma_n = 0$ if $\gamma_n \neq 2^p$, and in the case 2 $\gamma_n = 1$ if $n = 2^p + 1$ and $\gamma_n = 0$ if $n \neq 2^p + 1$ ($p = 1, 2, \ldots$).

We denote by Q the set of potentials $q(x)$ such that the system $\{u_n(x)\}$ is a Riesz basis for $L_2(0,1)$, $\bar{Q} = L_1(0,1) \setminus Q$. From Theorem 1 and Theorem 2 it is easy to obtain the following

Corollary. *The sets Q and $\bar{Q}$ are dense everywhere in $L_1(0,1)$.*

Convergence of spectral expansions corresponding to problem (1)+(2) was studied by O.A. Veliev, N.B. Kerimov, Kh.P. Mamedov.

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